

Transport theory for cold relativistic superfluids from an analogue model of gravity

Massimo Mannarelli and Cristina Manuel¹

¹¹ *Instituto de Ciencias del Espacio (IEEC/CSIC)*

*Campus Universitat Autònoma de Barcelona, Facultat de Ciències, Torre C5
E-08193 Bellaterra (Barcelona), Spain*

(Dated: February 5, 2008)

We write a covariant transport equation for the phonon excitations of a relativistic superfluid valid at small temperatures. The hydrodynamical equations for this system are derived from the effective field theory associated to the superfluid phonons. We describe how to construct the kinetic theory for the phonon quasiparticles using a relativistic generalization of the analogue model of gravity developed by Unruh. This gravity analogy relies on the equivalence between the action of a phonon field moving in a superfluid background with that of a boson propagating in a given curved space-time. Exploiting this analogy we obtain continuity equations for the phonon current, entropy and energy-momentum tensor in a covariant form, valid in any reference frame. Our aim is to shed light on some aspects of transport phenomena of relativistic superfluidity. In particular, we are interested in the low temperature regime of the color flavor locked phase, which is a color superconducting and superfluid phase of high density QCD that may be realized in the core of neutron stars.

PACS numbers: 47.75.+f, 97.60.Jd, 05.10.Gg, 12.38.t, 47.37.+q

I. INTRODUCTION

Superfluidity is a phenomenon that occurs in some systems at sufficiently low temperatures after the appearance of a quantum condensate [1, 2]. Landau gave an explanation of this effect and developed his famous two-fluid model considering that the system can be viewed as composed by a superfluid component, which describes the coherent motion of the condensate, and a normal fluid component, where dissipative processes are allowed. The property of superfluidity follows from the existence of some elementary excitations with a linear dispersion law, the phonons. These phonons are the Goldstone modes which originate from the spontaneous breaking of a global symmetry caused by the appearance of the condensate.

In the cold regime dissipative processes are mainly due to the collisions of phonons, which are part of the normal fluid component. The viscous hydrodynamics of a superfluid is involved because one can define more transport coefficients than in a normal fluid. However, for non-relativistic superfluids one can compute from the kinetic theory constructed by Khalatnikov [2] all the transport coefficients that enter into the two-fluid hydrodynamical equations.

Relativistic generalizations of Landau's two-fluid model of superfluidity have been developed for applications to neutron stars physics. These generalizations are rather non-trivial, and alternative formulations of the hydrodynamics have been proposed in the literature [3, 4, 5, 6, 7, 8]. In the non-dissipative limit it is possible to show that all these approaches are equivalent. Unfortunately, very little is known about dissipative effects in relativistic superfluids [9, 10, 11]. To the best of our knowledge, the whole set of dissipative terms which enter into the relativistic hydrodynamical equations has only been derived in one of the above mentioned approaches [9, 10].

It is our aim to formulate the kinetic theory associated to the phonons of a relativistic superfluid, as these quasiparticles should dominate the transport phenomena at low temperatures. While the dynamics of the phonons in the frame where the superfluid component is at rest is easily formulated, this is in principle not so when done in an arbitrary frame. Relativistic boosts of the different variables complicate in an unnecessary way the problem. However, there is still the possibility to obtain transport equations in a fully covariant way. The approach exploits the gravity analogs which are naturally associated to relativistic hydrodynamical waves [12], following ideas of Unruh [13] and others (see Ref.[14] for a review and references).

Let us recall here that Unruh established long ago an analogy between supersonic flow in hydrodynamics and black hole physics [13]. Exploiting this hydrodynamics/gravity analogy [15], Volovik pointed out that the kinetic equation of the phonons of a non-relativistic superfluid can be rewritten as that of a boson evolving in Unruh's acoustic metric [16]. It is thus natural to expect that the phonons of a relativistic superfluid admit a similar treatment, as we show here. Let us mention that Popov also derived a similar transport equation for the superfluid phonons [17]. We will implement the same ideas in a somewhat different way.

Relativistic superfluids may be realized in nature in the interior of neutron stars where the temperature is low and the energy scale of the particles is sufficiently high. In particular, in the inner crust of neutron stars the attractive interaction between neutrons can lead to the formation of a BCS condensate, the system then becoming superfluid. Moreover, if deconfined quark matter is present in the core of neutron stars it will very likely be in a color

superconducting phase [18]. Quantum Chromodynamics (QCD) predicts that at asymptotically high densities quark matter is in the color flavor locked phase (CFL) [19]. In this phase up, down and strange quarks pair forming a quark condensate that is antisymmetric in color and flavor indices. The order parameter breaks the baryonic number $U(1)_B$ symmetry spontaneously, and therefore CFL quark matter is a superfluid as well.

The main part of the present paper is dedicated to the CFL superfluid. The starting point of our analysis is the low energy effective theory of the CFL phase which is obtained from the equation of state of quark matter [20]. Then we show how at asymptotic high density it is possible to derive from the microscopic physics, from QCD, the hydrodynamical behavior of the CFL superfluid.

Our ultimate goal is having a well-defined formalism to derive all the transport coefficients associated to the CFL superfluid. Up to now only the shear viscosity and bulk viscosity associated to the phonon contribution of the CFL superfluid have been computed [21, 22], as well as the contribution to the bulk viscosity due to kaons [23], and an estimate of the thermal conductivity [24]. However there are other transport coefficients which are peculiar of superfluid systems that are still unknown. These are needed in order to derive the macroscopic behavior of a hypothetical compact star made of CFL quark matter, and single out possible signatures of quark matter in astrophysical scenarios.

This paper is organized as follows. In Section II we review the convective variational approach of relativistic superfluid hydrodynamics. In Section III we review the derivation of the hydrodynamical equations for the CFL superfluid phase of QCD and the low energy effective action for the superfluid phonons. In Section IV we present the kinetic theory for the phonons of a relativistic superfluid and derive the continuity equations for the phonon current, entropy and energy-momentum tensor. In Section V the transport equations are specialized to a local equilibrium state and the corresponding expressions of number density, entropy density and pressure are computed. We show that in the limit of small velocities these quantities agree with those of the phonons of a non-relativistic superfluid. We draw our conclusions in Section VI. In Appendix A, we review an hydrodynamical approach to relativistic superfluidity by Son. We will assume a physical Minkowski space-time metric (there is no real gravity field) and use the conventions $\eta^{\mu\nu} = (1, -1, -1, -1)$, with natural units $c = \hbar = k_B = 1$.

II. HYDRODYNAMICS OF RELATIVISTIC SUPERFLUIDS: CONVECTIVE VARIATIONAL APPROACH

There are different formulations of the hydrodynamical equations governing a relativistic superfluid [3, 4, 5, 6, 7, 8]. All of them were derived as relativistic generalizations of the equations of Landau's two-fluid model of non-relativistic superfluid dynamics [1, 2]. Here we review the convective variational approach of Carter [4]. It has been shown in Ref.[5] that this approach is equivalent, at least in the non-dissipative limit, to the potential variational approach developed by Lebedev and Khalatnikov [3]. We also present in Appendix A the formulation derived by Son, which is equivalent to Carter's formulation as well.

In the convective variational approach one defines a master thermodynamic function Λ , which is a scalar function of three independent scalar variables, namely $n^\rho n_\rho$, $s^\rho s_\rho$ and $n^\rho s_\rho$, where s^ρ is the entropy current vector and n^ρ is the (total) particle number vector. All the thermodynamic equations of the superfluid can be derived from Λ and by partial differentiation one finds

$$d\Lambda = \Theta_\rho ds^\rho + \mu_\rho dn^\rho, \quad (1)$$

that serves to define the so-called thermal momentum covector Θ_ρ and the vector μ_ρ .

By applying a Legendre-type transformation, one can define the thermodynamics in terms of the pressure

$$P = \Lambda - s^\rho \Theta_\rho - n^\rho \mu_\rho, \quad (2)$$

which is a function of the conjugate variables $\mu^2 = \mu_\rho \mu^\rho$, $z^2 = \mu_\rho \Theta^\rho$ and $\Theta^2 = \Theta^\rho \Theta_\rho$. Here μ is the chemical potential of the system, and Θ represents the temperature, which we will also denote by T . By partial differentiation one has

$$dP = s^\rho d\Theta_\rho + n^\rho d\mu_\rho, \quad (3)$$

and then one can express the vectors n^ρ and s^ρ as a function of μ^ρ and Θ^ρ as

$$n^\rho = F\mu^\rho + Q\Theta^\rho, \quad s^\rho = Q\mu^\rho + G\Theta^\rho, \quad (4)$$

where

$$F = 2 \frac{\partial P}{\partial \mu^2}, \quad Q = \frac{\partial P}{\partial z^2}, \quad G = 2 \frac{\partial P}{\partial \Theta^2}. \quad (5)$$

Finally the energy-momentum tensor of the system is given by

$$T^{\rho\sigma} = n^\rho \mu^\sigma + s^\rho \Theta^\sigma - P \eta^{\rho\sigma} . \quad (6)$$

Although the notation used does not demonstrate the symmetry of the tensor explicitly, it really takes place [5].

The hydrodynamical equations of the superfluid are given by

$$\begin{aligned} \partial_\rho n^\rho &= 0 , & \partial_\rho s^\rho &= 0 , \\ s^\rho \partial_{[\rho} \Theta_{\sigma]} &= 0 , & \partial_{[\rho} \mu_{\sigma]} &= 0 , \end{aligned} \quad (7)$$

where the brackets denote index antisymmetrization. The irrotational condition equation on μ_ρ tells that this vector can be expressed as the gradient of a scalar function,

$$\mu_\rho = -\partial_\rho \varphi . \quad (9)$$

As it will be shown in the following Section, for the CFL superfluid it is possible to show that this scalar function is related to the phase of the diquark condensate [20].

Using Eq. (6) and Eqs. (7) one obtains the energy-momentum conservation law

$$\partial_\rho T^{\rho\sigma} = 0 . \quad (10)$$

From the expressions above it is clear that the hydrodynamical equations keep the form of conservation laws, exactly as in a normal fluid.

In the zero temperature limit and for the ideal case of no interaction between the particles, the pressure is only a function of the chemical potential, and the entropy current vanishes. In this case, one defines the velocity vector

$$v^\rho = \frac{\mu^\rho}{\mu} , \quad (11)$$

such that it is properly normalized, $v^\rho v_\rho = 1$. In the $T = 0$ limit the energy-momentum tensor takes the form of that of an ideal fluid and then Eq. (6) can be written as

$$T^{\alpha\beta} = (n\mu) v^\alpha v^\beta - P \eta^{\alpha\beta} = (\rho + P) v^\alpha v^\beta - P \eta^{\alpha\beta} , \quad (12)$$

where ρ is the energy density of the system, and we have used the zero temperature relation $n\mu = \rho + P$.

Unfortunately in this formulation of the superfluid hydrodynamics it is not a priori obvious what is the physical meaning of all the hydrodynamical variables. The approach presented in this paper allows one to clarify their field-theoretical origin.

III. THE CFL SUPERFLUID IN THE LIMIT OF VANISHING TEMPERATURE

In this Section we study the relativistic superfluid system composed of color-flavor locked (CFL) quark matter. The CFL phase has long been known to be a superfluid: by picking a phase its order parameter breaks the baryonic number $U(1)_B$ symmetry spontaneously. At least in the limit of asymptotic high densities this is the energetically favored phase with respect to other less symmetric color superconducting phases and with respect to unpaired quark matter [18].

In the regime of sufficiently high densities it is possible to derive the hydrodynamical equations from the microscopic physics. Indeed in Ref. [20] Son showed how to obtain the effective action of the superfluid phonons by integrating out from the QCD Lagrangian the heavy degrees of freedom corresponding to gluons, quarks and mesons. At very high densities, this is an allowed operation because the only low energy excitations are the phonons. The classical equations of motion derived from this effective action correspond to the zero temperature hydrodynamical equations that can be obtained within the convective approach, discussed in Section II. This way of obtaining the Carter's equations allows one to identify the relation between the superfluid velocity and the gradient of the phase of the diquark condensate. Indeed, these two quantities turn out to be directly proportional.

In order to see thermal effects in the low temperature limit in full generality, one has to consider the dynamics of the phonons moving in the superfluid background. This can also be derived from Son's approach [22], and we discuss in Section IIIB how to obtain the zero temperature effective action of the phonons in the presence of the superfluid background. As announced in the Introduction, the equation of motion of the phonon field in the superfluid background can be interpreted as that of a scalar field moving in an "acoustic" non-flat metric. This means that

we will write the equation of motion in a covariant form with respect to such a non-flat metric. This formulation is rather useful, as then one can infer the phonon effective action in any frame, and not necessarily in the superfluid rest frame. We will then use it in order to obtain the transport theory in covariant form.

Let us stress here that we only consider the very low temperature regime, and that by increasing the temperature other quasiparticles can be excited in the CFL phase. Although such excitations might be relevant for transport phenomena at moderately higher temperature [23], for the sake of simplicity we will ignore them, as we assume a very low temperature.

A. The effective Lagrangian of the superfluid phonons

The effective field theory for the Goldstone boson originating from the spontaneous breaking of the $U(1)_B$ symmetry can be constructed from the equation of state (EoS) of normal quark matter, following the procedure of Ref. [20]. Because the superfluid phonon is a Goldstone boson, and thus a light degree of freedom, its low energy effective field theory is obtained by integrating out the heavy modes from the QCD Lagrangian. This operation amounts to a minimization procedure with respect to the modulus of the order parameter of the symmetry. Exploiting other symmetries of the problem, one can entirely fix the form of the phonon effective action from the knowledge of the pressure of CFL quark matter.

Calling φ the phase of the condensate, and defining $D_\rho\varphi \equiv \partial_\rho\varphi - \mu_q A_\rho$, with $A_\rho = (1, 0, 0, 0)$ and μ_q the quark chemical potential, the low energy effective Lagrangian for φ (valid for energy scales much smaller than the CFL gap Δ) is expressed as

$$\mathcal{L}_{\text{eff}}[D_\rho\varphi] = P[(D_\rho\varphi D^\rho\varphi)^{1/2}] . \quad (13)$$

In this expression P is a functional of the derivatives of the field φ that has the same form as the pressure of the system at zero temperature. Therefore in order to obtain the Lagrangian for φ is enough to specify P . At asymptotic large densities the EoS of CFL quark matter takes the form

$$P(\mu_q) = \frac{3}{4\pi^2} \mu_q^4 , \quad (14)$$

corresponding to a system of three flavors of massless quarks. The validity of such an expression relies on the fact that at very high μ_q the coupling constant is small $g(\mu_q) \ll 1$ and the effects of interactions and the effects of Cooper pairing are subleading. Therefore they are neglected in Eq. (14). One assumes that the quark masses give a subleading effect as well, because $m_q \ll \mu_q$. Effects due to the strange quark mass are considered in Ref. [22]. In the end one obtains from Eq. (14) that the low energy effective Lagrangian for the CFL phase in the limit of asymptotic densities is given by

$$\mathcal{L}_{\text{eff}} = \frac{3}{4\pi^2} [(\partial_0\varphi - \mu_q)^2 - (\partial_i\varphi)^2]^2 . \quad (15)$$

There is an interesting interpretation of the equations of motion associated to φ . Since the Lagrangian in Eq. (15) does not explicitly depend on the field φ , but only on its derivatives, the corresponding classical equation of motion takes the form of a conservation law. Then one can view it as the hydrodynamical conservation law of a current representing baryon number flow,

$$\partial_\nu(n_0 v^\nu) = 0 , \quad (16)$$

where

$$n_0 = \left. \frac{dP}{d\mu} \right|_{\mu=\bar{\mu}} = \frac{3}{\pi^2} \bar{\mu}^3 \quad (17)$$

is interpreted as the baryon density [20], where $\bar{\mu} = (D_\rho\bar{\varphi} D^\rho\bar{\varphi})^{1/2}$ and where

$$v_\rho = -\frac{D_\rho\bar{\varphi}}{\bar{\mu}} , \quad (18)$$

is the superfluid velocity with $\bar{\varphi}$ the solution of the classical equation of motion. Notice that the superfluid velocity is properly normalized, that is, $v_\rho v^\rho = 1$, and that in this definition it is taken into account that Lorentz symmetry is explicitly broken by μ_q .

The energy-momentum tensor can be written in terms of the velocity defined in Eq. (18) and Noether's energy-density ρ_0 ,

$$T_0^{\rho\sigma} = (n_0\bar{\mu})v^\rho v^\sigma - g^{\rho\sigma}P_0 = (\rho_0 + P_0)v^\rho v^\sigma - \eta^{\rho\sigma}P_0, \quad (19)$$

where we have written $\rho_0 + P_0 = n_0\bar{\mu}$, with P_0 the quark pressure evaluated at $\bar{\mu}$. The energy-momentum tensor is conserved

$$\partial_\rho T_0^{\rho\sigma} = 0, \quad (20)$$

and traceless $T_{0\rho}^\rho = 0$.

One can immediately check that Eqs. (16) and (20) are equivalent to the hydrodynamical equations of a relativistic superfluid at zero temperature obtained in the convective variational approach. The definition of the superfluid velocity differs in the zero component, but this can be matched by the change $\varphi \rightarrow \varphi + \mu x_0$.

B. Phonons moving in the superfluid background

From the expression of the Lagrangian in Eq. (15) it is possible to derive the effective field theory of the phonons moving in the background of the superfluid [22]. In the CFL phase the superfluid phonon is the Goldstone boson associated to the breaking of the $U(1)_B$ symmetry and it can be introduced as the phase of the diquark condensate. However, according to Eq.(18) the gradient of the phase of the condensate defines the superfluid velocity as well. Then it should be possible to decompose the phase of the condensate in two fields, the first describing the hydrodynamical variable, the second describing the quantum fluctuations associated to the phonons. Therefore in order to find the phonon dispersion relation in a moving superfluid we will consider the quantum fluctuations around the classical solution $\bar{\varphi}$ of the equations of motion associated to the Lagrangian (15), thus we write

$$\varphi(x) = \bar{\varphi}(x) + \phi(x). \quad (21)$$

This splitting implies a separation of scales - the background field $\bar{\varphi}(x)$ is associated to the long-distance and long-time scales, while the fluctuation $\phi(x)$ is associated to rapid and small scale variations. The gradient of $\bar{\varphi}(x)$ is proportional to the hydrodynamical velocity, according to Eq.(18), while the fluctuation ϕ is identified with the superfluid phonon.

From the low energy effective action of the system

$$S[\varphi] = \int d^4x \mathcal{L}_{\text{eff}}[\partial\varphi], \quad (22)$$

we deduce the effective action for the phonon field expanding around the stationary point corresponding to the classical solution

$$S[\varphi] = S[\bar{\varphi}] + \frac{1}{2} \int d^4x \frac{\partial^2 \mathcal{L}_{\text{eff}}}{\partial(\partial_\mu \varphi) \partial(\partial_\nu \varphi)} \Big|_{\bar{\varphi}} \partial_\mu \phi \partial_\nu \phi + \dots, \quad (23)$$

and considering the expression of the Lagrangian in Eq. (15) one has that

$$f^{\mu\nu} = \frac{\partial^2 \mathcal{L}_{\text{eff}}}{\partial(\partial_\mu \varphi) \partial(\partial_\nu \varphi)} \Big|_{\bar{\varphi}} = \frac{n_0}{\bar{\mu}} \left\{ \eta^{\mu\nu} + \left(\frac{1}{c_s^2} - 1 \right) v^\mu v^\nu \right\}. \quad (24)$$

Here $c_s = 1/\sqrt{3}$ is the speed of sound in CFL quark matter.

The second term on the right hand side of Eq.(23) represents the action of the linearized fluctuation - here the superfluid phonon - and can be written as the action of a boson moving in a non-trivial gravity background [14]

$$S[\phi] = \frac{1}{2} \int d^4x \sqrt{-\mathcal{G}} \mathcal{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (25)$$

where we have defined the metric tensor

$$\mathcal{G}^{\mu\nu} = \eta^{\mu\nu} + \left(\frac{1}{c_s^2} - 1 \right) v^\mu v^\nu \quad (26)$$

and the determinant $\mathcal{G} = 1/\det|\mathcal{G}^{\mu\nu}|$.

In writing the effective Lagrangian in Eq.(25) we have rescaled the phonon field according to

$$\phi \rightarrow \sqrt{\frac{c_s \bar{\mu}}{n_0}} \phi, \quad (27)$$

to get a dimensional scalar field, and thus a dimensionless metric in natural units. In doing the rescaling, we have assumed that on the scale of variations of ϕ , both $\bar{\mu}$ and n_0 can be assumed to be constant. This is justified by the separation of scales which is implicit in the splitting of the bosonic field φ done in Eq. (21).

The normalized field obeys the classical equation of motion

$$\partial_\mu \left(\sqrt{-\mathcal{G}} \mathcal{G}^{\mu\nu} \partial_\nu \phi \right) = 0. \quad (28)$$

We then see that from the effective Lagrangian in Eq. (15) one can derive in Eq. (26) the so-called sonic or acoustic metric tensor $\mathcal{G}^{\mu\nu}$ [6, 14] that depends on the speed of sound c_s and on the superfluid velocity v^μ . Note that since the superfluid velocity v^μ is normalized to 1, in general the metric tensor $\mathcal{G}^{\mu\nu}$ corresponds to a strong gravitational field. The limit of weak gravitational field is achieved for $c_s \rightarrow 1$, and only in this (unphysical) limit one can expand the gravitational metric tensor around the flat metric $\eta_{\mu\nu}$.

The dispersion law of phonons in the superfluid medium are obtained by solving the equation of motion for a massless particle moving in curved space time, that is

$$\mathcal{G}^{\mu\nu} p_\mu p_\nu = 0, \quad (29)$$

where $p_\mu = (E, -\mathbf{p})$. In the superfluid rest frame, i.e. where $v^\mu = (1, \mathbf{0})$, the dispersion law simplifies to the form

$$E = c_s p, \quad (30)$$

which expresses the fact that phonons propagate at the speed of sound. In a different frame the dispersion law takes a rather complicated form. In order to express such a relation in a compact way it is convenient to define the following Lorentz invariants [17]

$$\varepsilon = p_\mu v^\mu, \quad \pi^2 = -(\eta^{\mu\nu} - v^\mu v^\nu) p_\mu p_\nu, \quad (31)$$

which are the energy and the square of the 3-momenta in the local rest frame of the superfluid. It is easy to check from Eq.(29) that in any frame the relation $\varepsilon = c_s \pi$ holds.

IV. KINETIC THEORY FOR THE SUPERFLUID PHONONS

In order to study the contribution of the superfluid phonons to the hydrodynamics one has to consider a thermal bath of these quasiparticles, and compute their contribution to the various thermodynamical quantities. This analysis could be done in thermal field theory, using the action given in Eq. (25). Because we will be mainly interested in studying kinetic phenomena, and ultimately in computing transport coefficients, we find more convenient to develop a transport theory for the cold CFL superfluid. While it should be fully equivalent to the thermal field theory approach, it turns out that the computation of transport coefficients is much simpler when done with the use of kinetic theory, as it is the case for other field theories (see, e.g. [25]).

Transport equations can only be formulated when there are well-defined quasiparticles in the system, that is, thermal excitations that are sufficiently long lived. In the CFL superfluid a direct computation of the damping rate for the superfluid phonons shows that it is suppressed by $(T/\mu_q)^4$ with respect to their typical energy [21]. One can thus guarantee that it is possible to write a transport equation for these quasiparticles, as quark matter is in the CFL phase only when $T \ll \mu_q$.

A. Quasiparticle picture

As described in the previous Section, the dispersion law of the phonon moving in the superfluid background can be seen as that of a boson propagating on a gravity background characterized by the acoustic metric. In a classical approximation one can view the phonons as massless quasiparticles that follow the paths determined by the geodesics

of the acoustic metric, defined by $0 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu$. Moreover, it is possible to define a quasiparticle Hamiltonian, \mathcal{H} , in such a way that solving the Hamiltonian equations one obtains the geodesics of the problem. Thus, we introduce

$$\mathcal{H} = \frac{1}{2} \mathcal{G}^{\mu\nu} p_\mu p_\nu = \frac{1}{2} \mathcal{G}_{\mu\nu} p^\mu p^\nu = \frac{1}{2} p^\mu p_\mu, \quad (32)$$

where $p_\mu = \mathcal{G}_{\mu\nu} p^\mu$ and where $\mathcal{G}_{\mu\nu}$ is the inverse of the metric $\mathcal{G}^{\mu\nu}$, obtained by solving the equation $\mathcal{G}^{\mu\nu} \mathcal{G}_{\nu\rho} = \delta_\rho^\mu$. For the metric tensor defined in Eq.(26) the inverse is given by

$$\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + (c_s^2 - 1) v_\mu v_\nu, \quad (33)$$

where $v_\mu = \eta_{\mu\nu} v^\mu$.

The Hamiltonian equations of motion are given by

$$\frac{dx^\mu}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\mu} = p^\mu, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^\mu} = -\frac{1}{2} \partial_\mu \mathcal{G}^{\alpha\beta} p_\alpha p_\beta, \quad (34)$$

however, since we will formulate the transport approach using the variables (x^ρ, p^ρ) , rather than (x^ρ, p_ρ) , it is convenient to find the equation of motion for p^ρ . After simple algebraic manipulations one finds

$$\frac{dp^\rho}{d\tau} = -\mathcal{G}^{\rho\mu} (\mathcal{G}_{\mu\beta,\alpha} - \frac{1}{2} \mathcal{G}_{\alpha\beta,\mu}) p^\alpha p^\beta = -\Gamma_{\alpha\beta}^\rho p^\alpha p^\beta, \quad (35)$$

where $\Gamma_{\beta\gamma}^\alpha$ denote the Christoffel symbols associated to the metric $\mathcal{G}_{\mu\nu}$,

$$\Gamma_{\alpha\beta}^\rho = \frac{1}{2} \mathcal{G}^{\rho\mu} (\mathcal{G}_{\mu\beta,\alpha} + \mathcal{G}_{\mu\alpha,\beta} - \mathcal{G}_{\alpha\beta,\mu}). \quad (36)$$

With these Christoffel symbols one can define the covariant derivatives for a generic contravariant vector A^μ as

$$A_{;\nu}^\mu = A_{,\nu}^\mu + \Gamma_{\nu\alpha}^\mu A^\alpha, \quad (37)$$

where $A_{,\nu}^\mu = \partial_\nu A^\mu$. While for a covariant vector one has

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^\alpha A_\alpha. \quad (38)$$

B. Covariant transport equation for the superfluid phonons

We will now study the Liouville equation governing the evolution of the phonon distribution function $f(x, p)$. Here it is understood that $f(x, p) p^\mu n_\mu d\Sigma d\mathcal{P}$ is the number of particles whose world lines intersect the hypersurface element $n_\mu d\Sigma$ around x , having four-momenta in the range $(p, p + dp)$, with n_μ a four light-like vector that we will take as v_μ , and the momentum measure $d\mathcal{P}$ is chosen in such a way that it is coordinate invariant [26]

$$d\mathcal{P} = \sqrt{-\mathcal{G}} 2H(p) \delta(\mathcal{G}_{\mu\nu} p^\mu p^\nu) \frac{d^4 p}{(2\pi\hbar)^3}, \quad (39)$$

where $H(p) = 1$ if p is future oriented for an observer moving with velocity v_μ , and 0 otherwise.

The presence of the superfluid background will be manifest in the fact that the Liouville equation has to be written in a covariant form with respect to the metric $\mathcal{G}^{\mu\nu}$. Indeed, the evolution of the distribution function is governed by the equation

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} + \frac{\partial f}{\partial p^\alpha} \frac{dp^\alpha}{d\tau} = C[f], \quad (40)$$

where $C[f]$ is the collision term and upon using Eqs. (34,35) the Liouville equation can be rewritten as

$$L[f] \equiv p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \frac{\partial f}{\partial p^\alpha} = C[f] \quad (41)$$

that is the general relativistic version of the Boltzmann equation [26].

Knowing the distribution function one can construct the current, energy-momentum tensor and entropy associated to the phonons, which are given by the expressions

$$n_{\text{ph}}^\alpha = \int p^\alpha f(x, p) d\mathcal{P} , \quad (42)$$

$$T_{\text{ph}}^{\alpha\beta} = \int p^\alpha p^\beta f(x, p) d\mathcal{P} , \quad (43)$$

$$s_{\text{ph}}^\alpha = - \int p^\alpha [f \ln f - (1 + f) \ln (1 + f)] d\mathcal{P} . \quad (44)$$

Notice that in Section III B we explicitly made a splitting of fields, separating the low energy modes from the high energy modes, the last corresponding to the phonons. However, when writing the transport equation for the phonons one should keep in mind that the collective and thermal motion of the phonon fluid can occur at long scales. For this reason one has to keep the Christoffel symbols in the transport equation, which contain derivatives of the superfluid velocity. As it will be shown in Section V, the equilibrium distribution function of the phonons depends on $\mathcal{G}_{\mu\nu}$, meaning that all the above quantities evaluated at equilibrium vary on scales of the order of the scale of variation of the acoustic metric.

From the expressions of the current, energy-momentum tensor and entropy one can derive the continuity equations obeyed by these quantities. The analysis is quite simplified if one takes into account the following property [26]

$$\left[\int p^{\mu_1} \dots p^{\mu_l} p^\nu f(x, p) d\mathcal{P} \right]_{;\nu} = \int p^{\mu_1} \dots p^{\mu_l} L[f] d\mathcal{P} . \quad (45)$$

In the case of vanishing collision term, considering different values of l in this equation enables to obtain the various covariant conservation laws.

Taking $l = 0$ one has that the covariant continuity equation for the phonon number is given by

$$(n_{\text{ph}}^\nu)_{;\nu} = \left[\int p^\nu f(x, p) d\mathcal{P} \right]_{;\nu} = \int L[f] d\mathcal{P} = \int C[f] d\mathcal{P} , \quad (46)$$

which one can write explicitly as

$$\partial_\nu n_{\text{ph}}^\nu + \Gamma_{\mu\nu}^\mu n_{\text{ph}}^\nu = \int C[f] d\mathcal{P} , \quad (47)$$

and expresses the fact that collisional processes can violate phonon number. Moreover one has to consider interactions of the phonons with the superfluid background. This is reflected in the second term in the l.h.s of the above equation where a Christoffel symbol appears signaling that propagation of phonons is taking place in “curved” space-time. Since we can write

$$\Gamma_{\mu\alpha}^\mu = \frac{1}{\sqrt{-\mathcal{G}}} \partial_\alpha \sqrt{-\mathcal{G}} = \frac{1}{c_s} \frac{\partial c_s}{\partial x^\alpha} , \quad (48)$$

it follows that in the collisionless case, the variation of the number of phonons stems only from a possible space-time dependence of the speed of sound. Therefore in the collisionless case the covariant conservation law of the phonon current can be written in a compact form as

$$(n_{\text{ph}}^\nu)_{;\nu} = \frac{1}{\sqrt{-\mathcal{G}}} \partial_\alpha \left(\sqrt{-\mathcal{G}} n_{\text{ph}}^\alpha \right) = \frac{1}{c_s} \partial_\alpha (c_s n_{\text{ph}}^\alpha) = 0 . \quad (49)$$

Notice that in the absence of collisions the quantity that is conserved with respect to the flat metric, i.e. with respect to the derivative ∂_α , is the current

$$\tilde{n}_{\text{ph}}^\alpha = \sqrt{-\mathcal{G}} n_{\text{ph}}^\alpha . \quad (50)$$

This is the quantity that should be identified with the conserved phonon current, and obeys the continuity equation

$$\partial_\alpha \tilde{n}_{\text{ph}}^\alpha = 0 , \quad (51)$$

in the absence of collisions.

From Eq. (45) with $l = 1$, one obtains the covariant continuity equation for the energy-momentum tensor

$$(T_{\text{ph}}^{\mu\nu})_{;\nu} = \int p^\mu C[f] d\mathcal{P} = 0, \quad (52)$$

where we have assumed that collisions conserve both energy and momentum. The equation above can be written more explicitly as

$$(T_{\text{ph}}^{\mu\nu})_{;\nu} = (T_{\text{ph}}^{\mu\nu})_{,\nu} + \Gamma_{\nu\alpha}^\nu T_{\text{ph}}^{\mu\alpha} + \Gamma_{\nu\alpha}^\mu T_{\text{ph}}^{\nu\alpha} = 0, \quad (53)$$

that can be rewritten as follows,

$$\partial_\alpha \left(\sqrt{-\mathcal{G}} T_{\text{ph}}^{\mu\alpha} \right) + \sqrt{-\mathcal{G}} T_{\text{ph}}^{\nu\alpha} \Gamma_{\nu\alpha}^\mu = 0. \quad (54)$$

As in the case of the phonon current, it is useful to redefine the energy-momentum tensor as

$$\tilde{T}^{\mu\nu} = \sqrt{-\mathcal{G}} T^{\mu\nu}, \quad (55)$$

and in this way the four-momentum associated with it is given by

$$P^\nu = \int d^3x \sqrt{-\mathcal{G}} T^{0\nu} = \int d^3x \tilde{T}^{0\nu}. \quad (56)$$

Notice that defining the energy-momentum tensor of the phonons as $\tilde{T}^{\mu\nu}$, we have that the total conserved tensor of the system is given by

$$T^{\mu\nu} = T_0^{\mu\nu} + \tilde{T}_{\text{ph}}^{\mu\nu}, \quad (57)$$

where $T_0^{\mu\nu}$ is the component of the energy-momentum associated to the superfluid background.

If we assume that the collisional integral is zero, one can expect to have also the covariant conservation of entropy, i.e.

$$(s_{\text{ph}}^\alpha)_{;\alpha} = \frac{1}{\sqrt{-\mathcal{G}}} \partial_\alpha \left(\sqrt{-\mathcal{G}} s_{\text{ph}}^\alpha \right) = \frac{1}{\sqrt{-\mathcal{G}}} \partial_\alpha (\tilde{s}_{\text{ph}}^\alpha) = 0, \quad (58)$$

where $\tilde{s}_{\text{ph}}^\alpha = \sqrt{-\mathcal{G}} s_{\text{ph}}^\alpha$. In the presence of collisions this relation is not satisfied, and then there is entropy generation.

Let us remark on the meaning of the continuity equations of phonons that we have derived [16]. Since the phonon fluid moves on the top of the superfluid, even assuming that the collision term vanishes we still have the possibility that processes of interaction between the phonon fluid and the superfluid will lead to the exchange of momentum and energy among them. Indeed, even if energy and momentum of the whole system composed by the superfluid condensate and by the phonons is conserved, there can be energy and momentum exchanges between these two subsystems. This is also what happens when considering matter propagating in a gravity field, as also in that case the energy-momentum tensor associated to the “matter” field is not strictly conserved.

The possibility of exchanges of energy and momentum mentioned above is reflected in the continuity Eq. (54) that we rewrite in the compact form

$$\partial_\mu \tilde{T}_{\text{ph}\nu}^\mu = \frac{1}{2} \tilde{T}_{\text{ph}}^{\mu\rho} \partial_\nu \mathcal{G}_{\rho\mu}, \quad (59)$$

using the definition in Eq.(55). On the right hand side of this conservation law there is a term that couples the energy-momentum tensor of the phonons with the derivative of the gravitational field, expressing in a covariant form the interaction between the phonons and the underlying superfluid. As a result, energy and momentum can be exchanged between the phonons and the background if the speed of sound is space-time dependent and/or if the superfluid velocity is space-time dependent. This fact can be understood also within the gravitational analogy. Indeed, the energy-momentum tensor of any matter field, here the phonon field, couples with the gravitational field $\mathcal{G}_{\mu\nu}$ inducing exchange of energy and momentum between the gravitational and matter fields.

Let us finish this Section by stressing that the interesting point of the transport theory presented here is that the use of the gravitational analogy automatically allows to write the equations governing the variations of phonon currents and energy-momentum tensor in terms of covariant equations with respect to the acoustic metric.

V. THERMAL EQUILIBRIUM OF THE SUPERFLUID PHONONS

The covariant continuity equations for the phonons derived in the previous Section take a very simple form when the macroscopic variables associated to the superfluid are homogeneous or constant. Indeed in this case the Christoffel symbols vanish, i.e. $\Gamma_{\nu\alpha}^\mu = 0$, and covariant derivatives are equal to ordinary derivatives. Thus, the kinetic equation for the phonons are the same that one obtains for ordinary particles evolving in a flat metric. In this case one can find the equilibrium distribution function for the phonons in the exact same way as for ordinary particles.

One can as well find global equilibrium solutions for stationary situations (no time dependence). Global, or collisional equilibrium, distribution functions are given by

$$f_{\text{eq}}(x, p) = \frac{1}{\exp(p^\mu \beta_\mu) - 1}, \quad (60)$$

where β_μ is a vector that can be derived as follows. Since $L[f_{\text{eq}}] = 0$ one has that β^μ obeys the equation

$$(\beta_{\lambda,\rho} - \beta_\alpha \Gamma_{\lambda\rho}^\alpha) p^\lambda p^\rho = 0, \quad (61)$$

that can be rewritten as

$$\beta_{\lambda;\rho} + \beta_{\rho;\lambda} = 0. \quad (62)$$

Assuming that β_μ is time independent, it is possible to check that one possible solution to the above equations is given by

$$\beta^\mu = (1/T, \mathbf{0}), \quad (63)$$

where T is a constant that we identify with the temperature.

There are other local equilibrium configurations that are found by demanding no local entropy generation, that is $s_{;\alpha}^\alpha = 0$ and with solutions given by the collisional invariants of the theory. In this case we can expect that the equilibrium distribution function takes the form

$$f_{\text{eq}}(x, p) = \frac{1}{\exp(p^\mu \beta_\mu) - 1}, \quad (64)$$

where

$$\beta^\mu = \beta(x) u^\mu(x). \quad (65)$$

In the local equilibrium configuration the phonon energy-momentum tensor has to take the form

$$T_{\text{ph}}^{\mu\nu} = H u^\mu u^\nu - K \mathcal{G}^{\mu\nu}, \quad (66)$$

where H and K are two scalars. The reason for this form is that $T_{\text{ph}}^{\mu\nu}$ is a tensor and because of the definition in Eq. (43) it can be built by the tensors $u^\mu u^\nu$ and $\mathcal{G}^{\mu\nu}$ only. We will fix the scalars H and K as follows. When both the superfluid and phonon fluid are at rest, i.e. $u^\mu = v^\mu = (1, \mathbf{0})$, the pure temporal component of this tensor should give the phonon energy density at rest, while the spatial components are diagonal and proportional to the pressure

$$T_{\text{ph}}^{00} = \rho_{\text{ph}}, \quad T_{\text{ph}}^{ij} = P_{\text{ph}} \delta^{ij}. \quad (67)$$

Thus one can identify

$$H = \rho_{\text{ph}} + P_{\text{ph}}/c_s^2, \quad K = P_{\text{ph}}, \quad (68)$$

and the phonon energy-momentum tensor is then expressed as

$$T_{\text{ph}}^{\mu\nu} = (\rho_{\text{ph}} + P_{\text{ph}}/c_s^2) u^\mu u^\nu - P_{\text{ph}} \eta^{\mu\nu} - P_{\text{ph}} \left(\frac{1}{c_s^2} - 1 \right) v^\mu v^\nu. \quad (69)$$

From the definition in Eq. (43) one sees that the energy-momentum tensor is traceless, i.e. $\mathcal{G}_{\mu\nu} T_{\text{ph}}^{\mu\nu} = 0$. Upon substituting the expression of $T_{\text{ph}}^{\mu\nu}$ that we have obtained above, one has that the traceless condition leads to the equation

$$\rho_{\text{ph}} = \frac{P_{\text{ph}}}{c_s^2} \left(\frac{4c_s^2 - \mathcal{G}_{\mu\nu} u^\mu u^\nu}{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \right). \quad (70)$$

Moreover, using the definition of the energy-momentum tensor in Eq. (55) and the condition $\mathcal{G}_{\mu\nu}T_{\text{ph}}^{\mu\nu} = 0$, one finds

$$\tilde{T}_{\text{ph}}^{\mu\nu} = 4\tilde{P}_{\text{ph}}\hat{u}^\mu\hat{u}^\nu - \tilde{P}_{\text{ph}}\mathcal{G}^{\mu\nu}, \quad (71)$$

where we have defined

$$\tilde{P}_{\text{ph}} = \sqrt{-\mathcal{G}}P_{\text{ph}} \quad \text{and} \quad \hat{u}^\mu = \frac{u^\mu}{\sqrt{u^\mu u^\nu \mathcal{G}_{\mu\nu}}}. \quad (72)$$

We also define $\hat{u}_\mu = \mathcal{G}_{\mu\nu}\hat{u}^\nu$ and therefore $\hat{u}^\mu\hat{u}_\mu = 1$, meaning that \hat{u}^μ is a four-vector normalized to 1 with respect to the metric $\mathcal{G}_{\mu\nu}$.

It is important to note that the phonon contribution to the energy-momentum tensor keeps the form that is expected for the normal fluid component in a superfluid. Indeed, Eq. (69) has the tensorial structure which is expected in these systems according with the Poisson bracket approach to the hydrodynamics reviewed in the Appendix A (see Eq. (A2)).

In a similar way one finds that the expressions of the current and of the entropy current are of the form

$$\tilde{n}_{\text{ph}}^\mu = \tilde{n}_{\text{ph}}u^\mu, \quad \tilde{s}_{\text{ph}}^\mu = \tilde{s}_{\text{ph}}u^\mu, \quad (73)$$

where \tilde{n}_{ph} and \tilde{s}_{ph} are scalars.

Upon employing the definitions above in the Eqs. (42), (43) and (44), with the local equilibrium distribution function (64) one can compute the phonon current, entropy current and pressure. Then the energy density can be deduced from the pressure, according to Eq. (70). The best strategy for evaluating the integrals in Eqs. (42), (43) and (44) is to choose the frame where the superfluid is at rest $v_\mu = (1, \mathbf{0})$. Then, after defining $u^\mu = \gamma(1, \mathbf{u})$, where γ is the Lorentz factor, we obtain for $|\mathbf{u}| < c_s$

$$\tilde{n}_{\text{ph}} = \frac{T^3\zeta(3)}{\pi^2} \frac{1}{c_s^3\gamma^4(1 - \frac{\mathbf{u}^2}{c_s^2})^2}, \quad \tilde{s}_{\text{ph}} = \frac{T^3 2\pi^2}{45} \frac{1}{c_s^3\gamma^4(1 - \frac{\mathbf{u}^2}{c_s^2})^2}, \quad \tilde{P}_{\text{ph}} = \frac{T^4\pi^2}{90} \frac{1}{c_s^3\gamma^4(1 - \frac{\mathbf{u}^2}{c_s^2})^2}. \quad (74)$$

After realizing that in the superfluid rest frame

$$\mathcal{G}_{\mu\nu}u^\mu u^\nu = c_s^2\gamma^2(1 - \frac{\mathbf{u}^2}{c_s^2}), \quad (75)$$

one can express the same quantities in any arbitrary frame writing

$$\tilde{n}_{\text{ph}} = \sqrt{-\mathcal{G}} \frac{T^3\zeta(3)}{\pi^2} \frac{1}{(\mathcal{G}_{\mu\nu}u^\mu u^\nu)^2}, \quad \tilde{s}_{\text{ph}} = \sqrt{-\mathcal{G}} \frac{T^3 2\pi^2}{45} \frac{1}{(\mathcal{G}_{\mu\nu}u^\mu u^\nu)^2}, \quad \tilde{P}_{\text{ph}} = \sqrt{-\mathcal{G}} \frac{T^4\pi^2}{90} \frac{1}{(\mathcal{G}_{\mu\nu}u^\mu u^\nu)^2}. \quad (76)$$

From these expressions one can recover the non-relativistic results of Ref. [2]. Indeed, taking the limits $u^\mu \rightarrow (1, \mathbf{u}_{NR})$ and $v^\mu \rightarrow (1, \mathbf{v}_{NR})$, and using the same symbol for the non-relativistic speed of sound, one obtains that

$$\mathcal{G}_{\mu\nu}u^\mu u^\nu \rightarrow c_s^2 - (\mathbf{u}_{NR} - \mathbf{v}_{NR})^2, \quad (77)$$

and the expressions above turn into the relations

$$\tilde{n}_{\text{ph}} = \frac{T^3\zeta(3)}{\pi^2} \frac{1}{c_s^3(1 - \frac{\mathbf{w}^2}{c_s^2})^2}, \quad \tilde{s}_{\text{ph}} = \frac{T^3 2\pi^2}{45} \frac{1}{c_s^3(1 - \frac{\mathbf{w}^2}{c_s^2})^2}, \quad \tilde{P}_{\text{ph}} = \frac{T^4\pi^2}{90} \frac{1}{c_s^3(1 - \frac{\mathbf{w}^2}{c_s^2})^2}, \quad (78)$$

where $\mathbf{w} = \mathbf{u}_{NR} - \mathbf{v}_{NR}$ is the so-called counterflow velocity [2].

From the Eqs.(76) one can also verify that the thermodynamical relation

$$\tilde{s}_{\text{ph}}^\mu = \beta^\mu \tilde{P}_{\text{ph}} + \beta_\nu \tilde{T}_{\text{ph}}^{\mu\nu}, \quad (79)$$

is satisfied. In this relation indices are lowered/raised with the acoustic metric.

A similar expression for the phonon contribution to the pressure in a cold relativistic superfluid was found by Carter and Langlois [6]. The main difference of our treatment with that of Ref. [6] is that we assume that the velocity of the phonon fluid is relativistic. Carter and Langlois assumed a non-relativistic velocity for the phonon fluid, and accordingly, they obtained a non-relativistic relation among the thermodynamical variables instead of the relativistic one reported in Eq. (79).

For completeness we write the conservation law of the energy-momentum tensor in Eq.(59) in a more explicit form. Using the definition in Eq.(71) we can rewrite the left hand side of Eq.(59) as

$$\partial_\mu \tilde{T}_{\text{ph}\nu}^\mu = 4[(\partial_\mu \tilde{P}_{\text{ph}})\hat{u}^\mu \hat{u}^\nu + \tilde{P}_{\text{ph}}(\partial_\mu \hat{u}^\mu)\hat{u}^\nu + \tilde{P}_{\text{ph}}(\partial_\mu \hat{u}^\nu)\hat{u}^\mu] - \partial_\nu \tilde{P}_{\text{ph}}, \quad (80)$$

while the right hand side of Eq.(59) turns out to be

$$\frac{1}{2}\tilde{T}_{\text{ph}}^{\mu\rho}\partial_\nu\mathcal{G}_{\rho\mu} = \frac{1}{2}\tilde{P}_{\text{ph}}(4\hat{u}^\mu\hat{u}^\rho - \mathcal{G}^{\mu\rho})\partial_\nu\mathcal{G}_{\rho\mu}. \quad (81)$$

Projecting these equations along \hat{u}^ν we obtain that

$$4(\partial_\mu \tilde{P}\hat{u}^\mu) - \hat{u}^\mu\partial_\mu \tilde{P}_{\text{ph}} = \frac{\tilde{P}_{\text{ph}}}{2}(4\hat{u}^\mu\hat{u}^\rho - \mathcal{G}^{\mu\rho})\hat{u}^\nu\partial_\nu\mathcal{G}_{\rho\mu}. \quad (82)$$

Projecting in the directions orthogonal to \hat{u}^ν with $(\hat{u}^\alpha\hat{u}^\nu - \mathcal{G}^{\alpha\nu})$ one has

$$-4\tilde{P}_{\text{ph}}\mathcal{G}^{\alpha\nu}\hat{u}^\mu\partial_\mu\hat{u}_\nu - (\hat{u}^\alpha\hat{u}^\nu - \mathcal{G}^{\alpha\nu})\partial_\nu\tilde{P}_{\text{ph}} = \frac{\tilde{P}_{\text{ph}}}{2}(4\hat{u}^\mu\hat{u}^\rho - \mathcal{G}^{\mu\rho})(\hat{u}^\alpha\hat{u}^\nu - \mathcal{G}^{\alpha\nu})\partial_\nu\mathcal{G}_{\rho\mu}. \quad (83)$$

In the limit where the metric $\mathcal{G}_{\mu\nu}$ can be approximated with the flat metric $\eta_{\mu\nu}$, one has that $\hat{u}^\mu \rightarrow u^\mu$ and the equations above reproduce the hydrodynamical equations of a ultrarelativistic system.

VI. DISCUSSION

The properties of superfluids at low but non-vanishing temperature are influenced by the presence of phonons, the massless Goldstone bosons originating from the spontaneous breaking of a global symmetry. In case these quasiparticles are the only low energy degrees of freedom they will give the dominant contribution to the transport properties of the system in this cold regime.

The transport theory of a non-relativistic superfluid was developed long ago, see e.g. Ref. [2]. In the non-relativistic case the transport equations for the phonons can be expressed as those of a boson evolving in Unruh acoustic metric [16]. In the present paper we have extended this approach to relativistic superfluids. We have shown that in a relativistic superfluid the same gravity analogy can be employed for deriving the transport equations replacing Unruh's metric with a generalized relativistic acoustic metric. Similar results have been reported in Ref. [17], although implemented in a different manner. The advantage of this formulation is twofold. It allows us to obtain the continuity equations and to express the phonon hydrodynamical variables in a covariant form, valid in an arbitrary reference frame. Moreover, it clarifies the physical meaning of the variables used in the convective variational approach of the hydrodynamics.

The computation of transport coefficients can be implemented employing the framework presented in the present paper considering the Boltzmann equation (41) with a non-vanishing collision term. The relevant collision terms can be derived from the microscopic physics and depend on the particular system considered. For the CFL superfluid in the high density limit the collision term can be evaluated considering various scattering processes among superfluid phonons whose vertices can be read from the effective Lagrangian in Eq. (15). The leading processes are binary collisions, collinear splitting and joining processes [21]. The phonon contribution to the shear viscosity, and the bulk viscosity associated to the normal fluid component have been computed following this procedure in Refs. [21, 22]. These computations were done using kinetic theory in the superfluid rest frame, where one takes $v^\mu = (1, \mathbf{0})$, and assuming homogeneity in the superfluid flow. This corresponds to consider in our equations vanishing Christoffel symbols, $\Gamma_{\mu\nu}^\rho = 0$. The transport equation that we have presented will allow us to compute the viscosity coefficients that involve also gradients in the superfluid flow. The results of these computations will be reported soon.

Although we have focused our attention on the CFL superfluid the techniques developed here could be employed for different relativistic superfluids. In particular, it would be interesting to compute the phonon contribution to the different viscosity coefficients in the neutron superfluid that is supposed to be realized in the interior of neutron stars. The evaluation of all the bulk viscosity coefficients arising from non-equilibrium beta processes has been recently reported in Ref. [10]. It would be interesting to evaluate the phonon contribution to these viscosity coefficients. One should also consider that at densities achievable in the core of neutron stars the energetically favored phase might not be the CFL phase [18]. Among the various possible phases that can be realized one interesting possibility is that quark matter is in the Crystalline Color Superconducting phase [27], where the gap parameter is periodically modulated in space and therefore spontaneously breaks translational invariance. In this case the low energy degrees of freedom do not consist only of the phonon excitation associated to the spontaneous breaking of $U(1)_B$. Different

phonon excitations arising from the spontaneous breaking of space symmetries and describing the oscillation of the crystalline structure are part of the spectrum as well [28]. Moreover, gapless fermionic excitations are present and this makes the analysis of the low energy properties of this phase challenging. In such a situation the transport theory derived in the present paper should be extended to include all the remaining low energy modes.

Acknowledgments

We thank Felipe Llanes-Estrada for useful discussions. This work has been supported by the Ministerio de Educación y Ciencia (MEC) under grants AYA 2005-08013-C03-02 and FPA2007-60275.

APPENDIX A: POISSON BRACKET APPROACH TO THE HYDRODYNAMICS OF RELATIVISTIC SUPERFLUIDS

The hydrodynamical equations of a relativistic superfluid have been derived using the Poisson bracket (PB) method by Son in Refs. [7, 8]. In this approach one considers that the superfluid properties of a system arise from the spontaneous breaking of a continuous $U(1)$ symmetry, with the appearance of a Goldstone mode. Since hydrodynamics is an effective field theory valid at long time and long length scales, the standard fluid variables should couple to the Goldstone mode.

One defines the Hamiltonian of the system as a functional of the hydrodynamical variables and of the Goldstone field, and then postulates a set of PB among them to derive the hydrodynamical equations. The postulated brackets might be deduced from the canonical quantum commutators in the corresponding microscopic quantum theory. Some of them also follow from simple physical considerations.

The equation of state of the system can be deduced from the microscopic theory and one finds that the differential equation for the total pressure of the system is given by

$$dP = s_0 dT_0 + n_0 d\mu_0 + \frac{1}{2} V^2 d(\partial_\mu \varphi)^2, \quad (\text{A1})$$

where V is a parameter depending on the microscopic variables. As an example in complex scalar theories V is proportional to the expectation value of the scalar field [7, 8].

After a Legendre transformation of the pressure, one has that the energy density is given by $\rho = s_0 T_0 + n_0 \mu_0 - P$. Imposing Poisson brackets conditions on entropy density, particle density and momentum density one arrives at the hydrodynamical equations [7, 8]

$$\partial_\rho T^{\rho\sigma} = 0, \quad T^{\rho\sigma} = (\rho + P) u^\rho u^\sigma - P \eta^{\rho\sigma} + V^2 \partial^\rho \varphi \partial^\sigma \varphi \quad (\text{A2})$$

$$\partial_\rho (n_0 u^\sigma - V^2 \partial^\rho \varphi) = 0 \quad (\text{A3})$$

$$\partial_\mu (s_0 u^\mu) = 0 \quad (\text{A4})$$

$$u^\mu \partial_\mu \varphi + \mu_0 = 0. \quad (\text{A5})$$

In this formulation of the superfluid hydrodynamics there is a clearer interpretation of both the stress-energy tensor and current, as being due to the sum of the normal fluid part and the coherent motion of the condensate (the superfluid). Indeed both these quantities have contributions proportional to the hydrodynamical velocity u^μ , associated to the normal fluid, and to the parameter V . Notice that the entropy has only contributions due to the normal component, as it should be.

While in principle this formulation of the hydrodynamics looks very different from the one presented in Section II they are equivalent. Indeed it is possible to relate the various variables of the two approaches by means of the following equations [7, 8]

$$\Lambda = \rho + V^2 (\partial \varphi)^2, \quad \mu_\mu = -\partial_\mu \varphi, \quad (\text{A6})$$

$$s^\mu = s_0 u^\mu, \quad n^\mu = n_0 u^\mu - V^2 \partial^\mu \varphi, \quad (\text{A7})$$

$$\Theta_\mu = \frac{1}{s_0} [(T_0 s_0 + \mu_0 n_0) u_\mu + n_0 \partial_\mu \varphi]. \quad (\text{A8})$$

Even if the two approaches are very similar they differ in one subtle point. While in the PB approach the value of $u^\mu \partial_\mu \varphi$ is constrained, in the convective approach this is considered as a free parameter. Instead one fixes the norm of the gradient of the field φ , requiring that at $T = 0$ one has $\partial_\mu \varphi \partial^\mu \varphi = \mu^2$. Imposing the two normalizations

simultaneously over-constraints the system, reducing the degrees of freedom that the problem should have. Let us also comment that the quantity $\mu \neq \mu_0$. Indeed μ_0 should be considered as the chemical potential associated to the normal component of the superfluid evaluated in the comoving frame. Instead μ is the norm of the vector μ^ρ .

-
- [1] L. Landau and Lifschitz, “Fluid Mechanics” vol. 6, Prentice Hall, New Jersey.
 - [2] I. M. Khalatnikov, “Introduction to the Theory of Superfluidity”, Benjamin, New York, 1965.
 - [3] V. V. Lebedev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **56**, 1601 (1982); I. M. Khalatnikov and V. V. Lebedev, Phys. Lett. **91A**, 70 (1982).
 - [4] B. Carter, in *Journes Relativistes 1976*, edited by M. Cahen, R. Debever, and J. Geheniau (Universit Libre de Brussels, Brussels, 1976), pp 12-27; in *Journes Relativistes 1979*, edited by I. Moret-Bailly and C. Latremolire (Facult des Sciences, Anger, 1979), pp 166-182; in *A Random Walk in Relativity and Cosmology*, Proceedings of the Vadya-Raychaudhuri Festschrift, IARG, 1983, pp. 48-62.
 - [5] B. Carter and I. M. Khalatnikov, Phys. Rev. D **45**, 4536 (1992).
 - [6] B. Carter and D. Langlois, Phys. Rev. D **51**, 5855 (1995).
 - [7] D. T. Son, Int. J. Mod. Phys. A **16S1C**, 1284 (2001) [arXiv:hep-ph/0011246].
 - [8] D. T. Son, “Relativistic hydrodynamics of systems with spontaneous symmetry breaking: the Poisson bracket approach”, unpublished.
 - [9] C. Pujol and D. Davesne, Phys. Rev. C **67**, 014901 (2003) [arXiv:hep-ph/0204355].
 - [10] M. E. Gusakov, Phys. Rev. D **76**, 083001 (2007) [arXiv:0704.1071 [astro-ph]].
 - [11] M. A. Valle, Phys. Rev. D **77**, 025004 (2008) [arXiv:0707.2665 [hep-ph]].
 - [12] N. Bilic, Class. Quant. Grav. **16**, 3953 (1999) [arXiv:gr-qc/9908002].
 - [13] W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981); Phys. Rev. D **51** (1995) 2827.
 - [14] C. Barcelo, S. Liberati and M. Visser, Living Rev. Rel. **8**, 12 (2005) [arXiv:gr-qc/0505065].
 - [15] M. Stone, Phys. Rev. E **62**, 1341 (2000) [arXiv:cond-mat/9909315].
 - [16] G. E. Volovik, Phys. Rept. **351**, 195 (2001) [arXiv:gr-qc/0005091].
 - [17] V. Popov, Gen. Rel. Grav. **38**, 917 (2006) [arXiv:gr-qc/0607023].
 - [18] For reviews, see K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333; M. G. Alford, Ann. Rev. Nucl. Part. Sci. **51**, 131 (2001) [arXiv:hep-ph/0102047]; G. Nardulli, Riv. Nuovo Cim. **25N3**, 1 (2002) [arXiv:hep-ph/0202037]; S. Reddy, Acta Phys. Polon. B **33**, 4101 (2002) [arXiv:nucl-th/0211045]; T. Schäfer, arXiv:hep-ph/0304281; D. H. Rischke, Prog. Part. Nucl. Phys. **52**, 197 (2004) [arXiv:nucl-th/0305030]; M. Alford, Prog. Theor. Phys. Suppl. **153**, 1 (2004) [arXiv:nucl-th/0312007]; M. Buballa, Phys. Rept. **407**, 205 (2005) [arXiv:hep-ph/0402234]; H. c. Ren, arXiv:hep-ph/0404074; I. Shovkovy, arXiv:nucl-th/0410091; T. Schäfer, arXiv:hep-ph/0509068; M. Alford and K. Rajagopal, arXiv:hep-ph/0606157; M. G. Alford, A. Schmitt, K. Rajagopal and T. Schafer, arXiv:0709.4635 [hep-ph].
 - [19] M. G. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B **537**, 443 (1999) [arXiv:hep-ph/9804403].
 - [20] D. T. Son, arXiv:hep-ph/0204199.
 - [21] C. Manuel, A. Dobado and F. J. Llanes-Estrada, JHEP **0509**, 076 (2005) [arXiv:hep-ph/0406058].
 - [22] C. Manuel and F. Llanes-Estrada, JCAP **0708**, 001 (2007) [arXiv:0705.3909 [hep-ph]].
 - [23] M. G. Alford, M. Braby, S. Reddy and T. Schafer, Phys. Rev. C **75**, 055209 (2007) [arXiv:nucl-th/0701067].
 - [24] I. A. Shovkovy and P. J. Ellis, Phys. Rev. C **66**, 015802 (2002) [arXiv:hep-ph/0204132].
 - [25] D. F. Litim and C. Manuel, Phys. Rept. **364**, 451 (2002) [arXiv:hep-ph/0110104].
 - [26] R. W. Lindquist, Ann. Phys. **37**, 487 (1966); J. M. Stewart, “*Non-equilibrium Relativistic Kinetic Theory*”, Springer-Verlag, New York, 1971.
 - [27] M. G. Alford, J. A. Bowers and K. Rajagopal, Phys. Rev. D **63**, 074016 (2001) [arXiv:hep-ph/0008208]; J. A. Bowers, J. Kundu, K. Rajagopal and E. Shuster, Phys. Rev. D **64**, 014024 (2001) [arXiv:hep-ph/0101067]; J. A. Bowers and K. Rajagopal, Phys. Rev. D **66**, 065002 (2002) [arXiv:hep-ph/0204079]; K. Rajagopal and R. Sharma, Phys. Rev. D **74**, 094019 (2006) [arXiv:hep-ph/0605316].
 - [28] R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli, Phys. Lett. B **511**, 218 (2001) [arXiv:hep-ph/0101326]; R. Casalbuoni, E. Fabiano, R. Gatto, M. Mannarelli and G. Nardulli, Phys. Rev. D **66**, 094006 (2002) [arXiv:hep-ph/0208121]; M. Mannarelli, K. Rajagopal and R. Sharma, Phys. Rev. D **76**, 074026 (2007) [arXiv:hep-ph/0702021]; M. Mannarelli, K. Rajagopal and R. Sharma, AIP Conf. Proc. **964**, 264 (2007) [arXiv:0710.0331 [hep-ph]].